

A new criterion for zero quantum discord

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Abstract. We propose a new criterion to judge zero quantum discord for arbitrary bipartite states. A bipartite quantum state has zero quantum discord if and only if all blocks of its density matrix are normal matrices and commute with each other. Given a bipartite state with zero quantum discord, how to find out the set of local projectors, which do not disturb the whole state after being imposed on one subsystem, is also presented. A class of two-qubit X-state is used to test the criterion, and an experimental scheme is proposed to realize it. Consequently, we prove that the positive operator-valued measurement can not extinguish the quantum correlation of a bipartite state with nonzero quantum discord.

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1. Introduction

For a bipartite system prepared in an entangled state, a local measurement on one of the two subsystems will affect the other subsystem owing to the “nonlocal features” of the entanglement[1]. However, entanglement is not always necessary for illustrating the non-localities in a quantum system[2]. In 1998, a model of deterministic quantum computation with one qubit (DQC1) is proposed for quantum computing by using highly mixed states[3, 4], which has been experimentally implemented in 2008[5]. It is a good example illustrating that some highly mixed states, even fully separable, contain intrinsic quantum correlations, and have potential applications in the quantum computing. Furthermore, quantum correlation is found to be more robust than entanglement in a noisy environment, which makes the quantum algorithms based only on quantum correlation more robust than those based on entanglement[6, 7, 8].

If a bipartite quantum state is in a product state, $\rho = \rho_A \otimes \rho_B$, with ρ_A (ρ_B) being the reduced density matrix of subsystem A (B), the state has no quantum correlation. However, a state with zero quantum correlation is not always a product state. The quantum correlation of a bipartite state is usually measured by quantum discord, introduced by Ollivier and Zurek in ref.[9]. The question of how to find out whether a quantum state has zero quantum discord or not is fundamentally important; it is the first step to distinguish the quantum features of a bipartite state from the classical. For example, it is shown that zero quantum discord between a quantum system and its environment is necessary and sufficient for describing the evolution of the system through a completely positive map[10, 11]. In addition, a quantum state can be locally broadcasted if and only if it has zero quantum discord [12, 13]. Recently, a necessary and sufficient condition for nonzero quantum discord was proposed[14], with the help of a correlation matrix, derived from the density matrix, and its singular value decomposition. In this paper, we present a simpler method for judging zero quantum discord, where we only need to partition the density matrix into N^2 block matrices (N : the dimension of one subsystem), and check some properties of these block matrices. This method is valid for arbitrary bipartite states and easy to implement. An example with a scheme to experimentally realize it, is proposed in order to test this criterion. Based on this new criterion, we also prove that the positive operator-valued measurement (POVM) [15, 16] can not extinguish the quantum correlation of a bipartite state with nonzero quantum discord.

2. New criterion for zero quantum discord

Ollivier and Zurek introduced the concept of quantum discord to quantify the quantum correlation of a bipartite state, which is defined as the difference between two conditional entropies (classically equivalent quantities)[9],

$$\delta(\rho_{AB})_{\{|k_B\rangle\}} = H(A|\{|k_B\rangle\}) - [H(\rho_{AB}) - H(\rho_B)], \quad (1)$$

where $H(A|\{|k_B\rangle\})$ is calculated by $\sum_k p_{k_B} H(\rho_{k_B})$ with $\rho_{k_B} = \frac{1}{p_{k_B}} \langle k_B | \rho_{AB} | k_B \rangle$ and $p_{k_B} = \text{Tr}_A(\langle k_B | \rho_{AB} | k_B \rangle)$, and $H(\rho)$ is the von Neumann entropy of the quantum state ρ [17, 18, 19]. Here the subsystem A is regarded as the system and B as the apparatus. $\{|k_B\rangle\}$ represent a set of local projectors on B, rather than the POVM used in ref.[20]. In the calculation, different set of projectors will give out different values of quantum discord for the same quantum state. How to find out the set of local projectors which yields minimum quantum discord is very difficult[21, 22, 23, 24, 25, 26].

In a given basis, $\{|i_A k_B\rangle\}$ ($i = 1, 2, \dots, N$ and $k = 1, 2, \dots, M$), arranged as $\{|1_A 1_B\rangle, \dots, |1_A M_B\rangle, |2_A 1_B\rangle, \dots, |N_A M_B\rangle\}$, an AB bipartite quantum state can be described by the following density matrix,

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1(NM)} \\ \vdots & \ddots & \vdots \\ \rho_{(NM)1} & \cdots & \rho_{(NM)(NM)} \end{pmatrix}, \quad (2)$$

which has zero quantum discord if and only if it can also be written as[9]

$$\rho_{AB} = \sum_{i=1}^N \sum_{j=1}^N \sum_{k'=1}^M C_{i_A j_A k'_B} (|i_A\rangle\langle j_A|)(|k'_B\rangle\langle k'_B|), \quad (3)$$

with $C_{i_A j_A k'_B}$ being real or complex numbers and $\{|k'_B\rangle\}$ ($k' = 1, 2, \dots, M$) being a particular set of local projectors on B. The quantum state in the form of Eq.(3) is called pointer state[9], in which one can locally access the information in the system without changing the whole density matrix. Since the evaluation of quantum discord is asymmetric, and depends on which subsystem is chosen as the system and which one is the apparatus, the zero quantum discord of the quantum state (3) with the subsystem B being the apparatus, does not guarantee the zero quantum discord for A being the apparatus.

The $(NM) \times (NM)$ matrix in Eq.(2) can be partitioned into N^2 blocks,

$$\rho_{AB} = \begin{pmatrix} \rho^{(1_A 1_A)} & \cdots & \rho^{(1_A N_A)} \\ \vdots & \ddots & \vdots \\ \rho^{(N_A 1_A)} & \cdots & \rho^{(N_A N_A)} \end{pmatrix} \quad (4)$$

with each block being an $M \times M$ matrix,

$$\rho^{(i_A j_A)} = \begin{pmatrix} \rho^{((i-1)M+1)((j-1)M+1)} & \cdots & \rho^{((i-1)M+1)(jM)} \\ \vdots & \ddots & \vdots \\ \rho^{(iM)((j-1)M+1)} & \cdots & \rho^{(iM)(jM)} \end{pmatrix}, \quad (5)$$

which means the state(2) or (4) is equivalent to,

$$\rho_{AB} = \sum_{i=1}^N \sum_{j=1}^N (|i_A\rangle\langle j_A|) \rho^{(i_A j_A)}. \quad (6)$$

We rewrite the quantum state Eq.(3) in the basis $\{|i_A k_B\rangle\}$,

$$\rho_{AB} = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^M C_{i_A j_A k_B} (|i_A\rangle\langle j_A|) U (|k_B\rangle\langle k_B|) U^\dagger. \quad (7)$$

where the local unitary transformation U connects $\{|k_B\rangle\}$ and $\{|k'_B\rangle\}$ through the relation $|k'_B\rangle = U|k_B\rangle$. In order to make Eq.(6) have the same form of Eq.(7), all block matrices $\rho^{(i_A j_A)}$ must be able to be diagonalized by the same unitary transformation U ,

$$\rho^{(i_A j_A)} = U \left[\sum_{k=1}^M C_{i_A j_A k_B} (|k_B\rangle\langle k_B|) \right] U^\dagger, \quad (8)$$

which gives us the following relation,

$$\left[\rho^{(i_A j_A)}, (\rho^{(i_A j_A)})^\dagger \right] = 0. \quad (9)$$

The matrix satisfying Eq. (8) or (9) is called normal matrix[27]. In addition, since all $\rho^{(i_A j_A)}$ are diagonalized by the same unitary matrix U ,

$$\rho^{(i_A j_A)} = U \Lambda^{(i_A j_A)} U^\dagger, \quad (10)$$

they have the same eigen-vectors. Any two normal matrices have the same eigenvectors if and only if they commute with each other[27]. Consequently, we can conclude the criterion for zero quantum discord now: all N^2 blocks $\rho^{(i_A j_A)}$ in Eq.(4) are normal matrices (satisfying Eq.(9)), and must commute with each other. This criterion has the advantage that we can directly work on the matrix Eq.(2) in any tensor product basis, without the need to find out the particular basis, $\{|k'_B\rangle\}$, required in the criterion of Eq.(3). As we all known, a bipartite state with zero quantum discord, see Eq.(3), must be a separable state. Based on the new criterion, we can conclude that if a density matrix is composed of diagonal block matrices, it represents a separable state, which is valid for bipartite systems in any dimension. However, inverse case is not true. A separable state does not necessarily to have a density matrix composed of diagonal block matrices. Given a high dimensional bipartite or multipartite quantum state, how to efficiently verify its separability or entanglement is still an open question.

We stress here that although commutation relations are used to describe the criterion of zero quantum discord, just as done in ref.[14], the present criterion has no direct connection with that in ref.[14]. The number of commuting matrices used in ref.[14], denoted as L , is equal to the rank of the correlation matrix, which is smaller than or equal to the minimal one between the two squared dimension degrees of the two subsystems A and B, i.e., $L \leq \min\{N^2, M^2\}$. However, The number of commuting matrices used in our criterion is fixed as N^2 , with N being the dimension of the subsystem A. Furthermore, all the commuting matrices used in ref.[14] are Hermitian operators, while the commuting matrices in our criterion can be non-Hermitian or Hermitian, which depend on the density matrix itself.

If a quantum state ρ_{AB} has been verified to have zero quantum discord, we can obtain U and $\{|k'_B\rangle\}$ by diagonalizing any non-zero one of the block matrices $\rho^{(i_A j_A)}$ in Eq.(5). From Eqs. (6) and (10), we have

$$\rho_B = \text{Tr}_A(\rho_{AB}) = \sum_{i=1}^N \rho^{(i_A i_A)} = \sum_{i=1}^N U \Lambda^{(i_A i_A)} U^\dagger = U D U^\dagger, \quad (11)$$

with the diagonal matrix $D = \sum_{i=1}^N \Lambda^{(i_A i_A)}$, which tells us that the reduced matrix ρ_B can also be diagonalized by the unitary matrix U .

Let us now consider an example. Given a class of two-qubit X-state in the basis of $\{|1_A 1_B\rangle, |1_A 2_B\rangle, |2_A 1_B\rangle, |2_A 2_B\rangle\}$,

$$\rho_x = \begin{pmatrix} x & 0 & 0 & \sqrt{x(0.5-x)} \\ 0 & 0.5-x & \sqrt{x(0.5-x)} & 0 \\ 0 & \sqrt{x(0.5-x)} & x & 0 \\ \sqrt{x(0.5-x)} & 0 & 0 & 0.5-x \end{pmatrix}, \quad (x \in [0, 0.5]), \quad (12)$$

we can check whether the above states have zero quantum discord through three steps.

(1) Partition the density matrix (12) into four blocks:

$$\rho^{(1_A 1_A)} = \rho^{(2_A 2_A)} = \begin{pmatrix} x & 0 \\ 0 & 0.5-x \end{pmatrix}, \quad \rho^{(1_A 2_A)} = \rho^{(2_A 1_A)} = \begin{pmatrix} 0 & \sqrt{x(0.5-x)} \\ \sqrt{x(0.5-x)} & 0 \end{pmatrix}. \quad (13)$$

(2) Check whether the four blocks are normal matrices (satisfying Eq.(9)): Yes here.

(3) Check whether all of them commute with each other: as

$$\rho^{(1_A 1_A)} \rho^{(1_A 2_A)} = \begin{pmatrix} 0 & x\sqrt{x(0.5-x)} \\ (0.5-x)\sqrt{x(0.5-x)} & 0 \end{pmatrix} \quad (14a)$$

and

$$\rho^{(1_A 2_A)} \rho^{(1_A 1_A)} = \begin{pmatrix} 0 & (0.5-x)\sqrt{x(0.5-x)} \\ x\sqrt{x(0.5-x)} & 0 \end{pmatrix}, \quad (14b)$$

the equality $\rho^{(1_A 1_A)} \rho^{(1_A 2_A)} = \rho^{(1_A 2_A)} \rho^{(1_A 1_A)}$ holds true only when $x = 0, 0.25$ or 0.5 .

In the case of $x = 0.25$, the unitary transformation $U = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ diagonalizes the four matrices in Eq.(13), and the local projectors $\{|k'_B\rangle\}$ in the pointer state are $|1'_B\rangle = \frac{\sqrt{2}}{2}(|1_B\rangle + |2_B\rangle)$ and $|2'_B\rangle = \frac{\sqrt{2}}{2}(|1_B\rangle - |2_B\rangle)$. For $x = 0$ and 0.5 , the four matrices in Eq.(13) are already diagonal(or zero matrix), and $\{|1_B\rangle, |2_B\rangle\}$ is just the set of local projectors used in the pointer state.

The quantum discord of this state can directly be calculated by using the results in refs.[21, 22, 23, 24, 25, 26], which is

$$\delta(\rho_x) = -1 - (2x)\text{Log}_2(2x) - (1-2x)\text{Log}_2(1-2x) - \sum_{k=1}^2 [0.5 + (-1)^k \sqrt{2x(1-2x)}] \text{Log}_2[0.5 + (-1)^k \sqrt{2x(1-2x)}]. \quad (15)$$

The zero quantum discord occurs only when $x = 0, 0.25$ or 0.5 , which is the same as predicted by using our criterion.

3. Proposed experiment and discussions

Now we propose an experimental scheme to test the criterion based on the above X-state (12), which can be generated through the following procedure. The entangled photon pairs from the type-I parametric down conversion are in the state,

$$|\psi_1\rangle = \cos\theta|H_A H_B\rangle + \sin\theta|V_A V_B\rangle, \quad (16)$$

where θ is the angle of the pump polarization direction with respect to the vertical orientation, and the two optical axis of the non-linear crystals are arranged in horizontal and vertical orientations (H and V)[28, 29], respectively, see Fig. 1. An electro-optical modulator (EOM) in path A, switched on (acting as a half-wave plate) or off (performing nothing) through the control of a random number generator (RNG), convert, with probability 50%, the polarization of the photon in path A from V to H, or vice versus[30]. The quantum state after the EOM is,

$$\rho_{AB} = 0.5|\psi_1\rangle\langle\psi_1| + 0.5|\psi_2\rangle\langle\psi_2|, \quad (17a)$$

with

$$|\psi_2\rangle = \cos\theta|V_A H_B\rangle + \sin\theta|H_A V_B\rangle. \quad (17b)$$

The density matrix of the above state ρ_{AB} in the basis $\{|H_A H_B\rangle, |H_A V_B\rangle, |V_A H_B\rangle, |V_A V_B\rangle\}$ is just the X-state (12) with $x = 0.5\cos^2\theta$, which can be experimentally measured through the two-qubit tomography[31]. Now we have all the four block matrices, and we can apply them in our criterion to tell whether the quantum discord is zero or not. In order to verify the zero quantum discord for $x = 0.25$ and non-zero quantum discord for $x \neq 0, 0.25, 0.5$ experimentally, we use the following procedure. A half wave plate (HWP) in path B rotates the polarization of the photon in this path by the angle of 45° , which corresponds to the unitary transformation $U = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, as mentioned above. The H and V photons in this path will be separated by the polarization beam splitter (PBS) and then register on the two detectors D_1 and D_2 , respectively, which corresponds to two local orthogonal projectors on the photon B.

Once the photon B is detected by D_1 (with probability $p_1 = 2x$), or by D_2 (with probability $p_2 = 1 - 2x$), the other photon in path A will turn to the state ρ_1^A or ρ_2^A , accordingly, which can be found out through the single-qubit tomography[31]. With the measured ρ_1^A and ρ_2^A , we can construct a density matrix for the AB system,

$$\rho_{measure} = p_1 \rho_1^A \otimes (|\phi_1\rangle\langle\phi_1|) + p_2 \rho_2^A \otimes (|\phi_2\rangle\langle\phi_2|), \quad (18)$$

where $|\phi_1\rangle = \frac{\sqrt{2}}{2}(|H_B\rangle + |V_B\rangle)$ and $|\phi_2\rangle = \frac{\sqrt{2}}{2}(|H_B\rangle - |V_B\rangle)$ are the two eigenvectors associated with the measurement in current experimental setup. For $x = 0.25$, Eq.(18) is equal to equation (12), which means zero quantum discord. For $x \neq 0, 0.25, 0.5$, Eq.(18) will not be equal to the state of equation (12), no matter what kind of local projectors on the photon B are chosen, which means non-zero quantum discord. For the two trivial cases of $x = 0$ or 0.5 , the zero quantum correlation of the state (12) can be verified via the same method as above by removing the HWP in path B.

A useful result can be derived from our criterion: any POVM can't extinguish the quantum correlation of the bipartite state with nonzero quantum discord. For a bipartite quantum state ρ_{AB} , we do a POVM on subsystem B by attaching an ancillary system ρ_C on it, and making a projective measurement on the extended BC system. The new bipartite state, composed of one subsystem A and another subsystem B plus C, is $\rho_{A(BC)} = \rho_{AB} \otimes \rho_C$, which has zero quantum discord if and only if the quantum discord of the original bipartite quantum state ρ_{AB} is zero, no matter what kind of ancillary system is chosen. The matrix product rule[32] directly gives us the following equality,

$$\left[\rho^{(i_A j_A)} \otimes \rho_C, \rho^{(i'_A j'_A)} \otimes \rho_C \right] = \left[\rho^{(i_A j_A)}, \rho^{(i'_A j'_A)} \right] \otimes (\rho_C \rho_C) \quad (19)$$

and thus the above statement can be easily proven through the commutation relations among the blocks of the original density matrix ρ_{AB} , and also of the density matrix $\rho_{A(BC)}$. Therefore, the above criterion for zero quantum discord is valid for all types of local measurement, including POVM.

4. conclusions

To summarize, we derive a new criterion for zero quantum discord of arbitrary bipartite states, which is easy to be implemented with three steps: (1) Partition the density matrix of the $N \otimes M$ quantum state into N^2 block matrices; (2) Check whether every block is a normal matrix (commuting with its Hermitian transpose); (3) Check whether all block matrices commute with each other. For a bipartite state with zero quantum discord, we can find out the set of projectors, which do not change the whole state after being imposed on one of the subsystems, by diagonalizing any non-zero block matrix of its density matrix. This set of projectors provides a way to locally access the information in the system without disturbing the whole state. A class of two-qubit X-state is used to test the criterion, which can be experimentally implemented. It is also

shown that POVM can't extinguish the quantum correlation of a bipartite state with nonzero quantum discord, although it may have an effect on the evaluation of non-zero quantum correlation.

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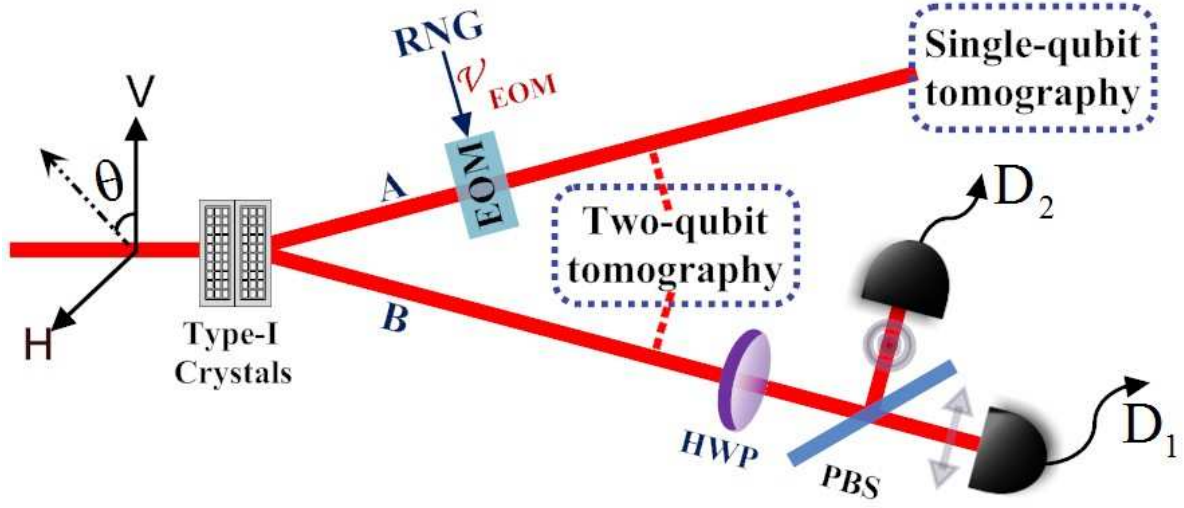


Figure 1. Proposed experimental setup: one of the two entangled photons (B), produced from non-linear crystals via type-I parametric down conversion, is sent to two single-photon detectors, D_1 and D_2 , after its polarization is rotated by a half wave plate (HWP). The polarization beam splitter (PBS) is used to distinguish the two types (H or V) of polarization of the photon B, and separate them. The polarization of the other photon (A) is inverted, with probability 50%, by the electro-optical modulator (EOM) and then measured through single-qubit tomography. The quantum state of the two photons after the EOM can be measured through two-qubit tomography.